III. Concyclic Points on an Equilateral Hyperbola and on its Inverses.

By R. M. Mathews, Wesleyan University.

The equation of an equilateral hyperbola may be written

$$xy - x - y = 0, \tag{1}$$

where the origin is a vertex and the axes of coördinates are parallel to the asymptotes. Writing $y = mx$, we can obtain the parametric equations

$$x = \frac{1+m}{m}, \qquad y = 1 + m. \tag{2}$$

There is one to one correspondence between all finite values of m (except zero) and all finite points on the curve.

These expressions (2) substituted in the equation of an arbitrary circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{3}$$

lead to the quartic equation

$$m^4 + 2(1+f)m^3 + (2 + 2g + 2f + c)m^2 + 2(1+g)m + 1 = 0. \tag{4}$$

The roots of this equation are the parameters of the points of intersection of the circle with the hyperbola. It is easy to show that the necessary and sufficient condition that four points of an equilateral hyperbola (2) be concyclic is that their parameters be such that

$$m_1m_2m_3m_4 = 1.$$

Take three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ of parameters k_i, l_i, m_i, respectively ($i = 1, 2, 3, 4$). The circle through A_i, B_i, C_i will cut the hyperbola in a fourth point D_i (parameter n_i) such that

$$k_il_im_in_i = 1 \qquad (i = 1, 2, 3, 4).$$

From these equations, with the help of the facts that the sets of A's, B's and C's are severally concyclic, we obtain

$$n_1n_2n_3n_4 = 1.$$

Thus we have proved the theorem:

Given three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ ($i = 1, 2, 3, 4$) on an equilateral hyperbola; let the circle through $A_iB_iC_i$ cut the curve in a fourth point D_i; then the four points D_i are concyclic.

As specializations of this proposition we have two known theorems, the first due to Cazamian.[1]

Two circles meet an equilateral hyperbola in $\{A_i\}$ and $\{C_i\}$ ($i = 1, 2, 3, 4$),

[1] "Sur l'hyperbole équilatère et sur ses inverses," *Nouvelles Annales de Mathématiques*, series 3, vol. 13, 1894, pp. 265–280.

respectively; draw the circle through A_i *and tangent to the curve at* C_i*; the four circles so obtained meet the conic again in four concyclic points.*

The four circles which osculate an equilateral hyperbola in four concyclic points cut the curve again in four concyclic points.

When an equilateral hyperbola is inverted about a point not on it, the transform is a bicircular quartic with a singular point at the center of inversion and the tangents there are at right angles (for they are parallel to the asymptotes of the conic). Thus the quartic is *orthotomic*. In particular, when the point of inversion is at the center of the hyperbola, the transform is a lemniscate. If the hyperbola be inverted about a point on it, the transform is a strophoid, oblique, in general, but a right strophoid, or Booth's logocyclica, when the center of inversion is a vertex of the conic. Every orthotomic bicircular quartic and every strophoid may be obtained by such an inversion.

As an inversion carries circles into circles, in general, the theorems just stated hold true when "equilateral hyperbola" is replaced by "orthotomic bicircular quartic" or by "strophoid," and the circles are conditioned as not passing through the node.

When one of the circles of the hyperbola passes through the center of inversion, special theorems result after the inversion. For example, in the main theorem for a strophoid let A_1 be at ∞, and then:

Let A_2*,* A_3*,* A_4 *be three collinear points of a strophoid while* $\{B_i\}$ *and* $\{C_i\}$ $(i = 1, 2, 3, 4)$ *are two sets of concyclic points. Let the line* B_1C_1 *cut the curve in* D_1*, while the circle on* $A_iB_iC_i$ $(i = 2, 3, 4)$ *meets the cubic in* D_i*. Then the four points* $\{D_i\}$ *are concyclic.*

In the case of one of the quartics the initial points may be arranged as: (1) one set of four collinear points with two sets of concyclic; (2) two sets of collinear points with one concyclic; (3) three sets of collinear points. None of the lines or circles of the quartic may pass through the singular point.

It is evident that the main theorem carries on to curves of still higher degree obtained by successive inversions.

On returning to the original theorem, we observe that there is no order assigned for the choice of a point triple to determine an (ABC) circle. When the three sets of concyclic points are assigned, it is evident that there are 64 circles (ABC) possible; each is determined by a triple of points, one from each set. Accordingly, there are 64 D-points which are concyclic in groups of four. To find how many of the possible ${}_{64}C_4$ are admissible here we observe: first, that each D-set is determined by a certain four of (ABC) circles; second, that in this set must be one circle for each of the A's; third, under this line of A's must be written some permutation of the four B's and some permutation of the four C's. Thus there are $\lfloor 4 \cdot \lfloor 4 = 576$ such sets and so 576 D-circles.

There are now $12 + 64 = 76$ points concyclic by fours in at least $576 + 64 + 3 = 643$ ways. It is easy to show that if a set of D-points be used with the B's and C's, the original A-set will be reached only once. It remains, then, to determine the conditions under which the configuration is closed as it stands, or will be closed after the addition of other points.